



# Some Useful Techniques

Solutions

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## 0S.1 ORDINARY DERIVATIVES

### 0S.1.1 Differentiation

#### Solution to Exercise 1

- (i)  $\sin' = \cos$  (see *Unit M100 12*) and so

$$\frac{d}{dx}(\sin x) = \boxed{\cos x}.$$

- (ii) The composite function rule is

$$(f \circ g)' = (f' \circ g) \times g'$$

(see *Unit M100 12*). Here  $f = \exp$  and  $g$  is  $x \mapsto (x - a)$ , or just  $(x - a)$  for short. Omitting the small circles as well, for brevity, we get

$$\begin{aligned} (\exp(x - a))' &= \exp'(x - a) \times (x - a)' \\ &= \exp(x - a) \times 1, \end{aligned}$$

since  $\exp' = \exp$  (see *Unit M100 12*),

$$= \exp(x - a).$$

We could alternatively have written the composite function rule in Leibniz notation,

$$\frac{dz}{dx} = \frac{dz}{dy} \frac{dy}{dx},$$

(often called the *chain rule* for functions of one variable), where  $z(y) = \exp y$  and  $y(x) = x - a$ . Then

$$\begin{aligned} \frac{dz}{dx} &= \exp y \times 1 \\ &= \exp(x - a). \end{aligned}$$

In either case we get

$$\frac{d}{dx}(\exp(x - a)) = \boxed{\exp(x - a)}.$$

- (iii) We can use either of the methods in part (ii) above. Using the Leibniz chain rule with  $z(y) = \sqrt{y}$  and  $y(x) = \sin x$ ,

$$\frac{dz}{dx} = \frac{1}{2}y^{-\frac{1}{2}} \times \cos x,$$

so that

$$\frac{d}{dx}(\sqrt{\sin x}) = \boxed{\frac{\cos x}{2\sqrt{\sin x}}}.$$

- (iv) The definitions we need are

$$\cosh x = \frac{1}{2}(e^x + e^{-x}), \quad \sinh x = \frac{1}{2}(e^x - e^{-x}),$$

from which we see that the derived function of  $\cosh$  is  $\sinh$  (not minus  $\sinh$ ; the analogy between trigonometric and hyperbolic functions is imperfect). Applying the chain rule as above with  $z(y) = \cosh y$ ,  $y(x) = x^2$ ,

$$\frac{dz}{dx} = \sinh y \times 2x,$$

so that

$$\frac{d}{dx}(\cosh x^2) = \boxed{2x \sinh x^2}.$$

### Solution to Exercise 2

Using the chain rule with  $z(y) = 1/y$ ,  $y(x) = \cos x$ ,

$$\begin{aligned}\frac{dz}{dx} &= -\frac{1}{y^2} \times (-\sin x), \\ &= \frac{\sin x}{\cos^2 x}\end{aligned}$$

so that

$$\frac{d}{dx}(\sec x) = \boxed{\tan x \sec x}.$$

If  $h = f/g$ , then

$$h' = \frac{f'g - fg'}{g^2}.$$

This is the form of the quotient rule stated in *Unit M100 12*. In Leibniz notation, if  $u = u(x)$  and  $v = v(x)$  then

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v \frac{du}{dx} - u \frac{dv}{dx}}{v^2}.$$

Putting  $u(x) = \sin x$ ,  $v(x) = \cos x$ ,

$$\begin{aligned}\frac{d}{dx}\left(\frac{\sin x}{\cos x}\right) &= \frac{\cos x(\cos x) - \sin x(-\sin x)}{\cos^2 x} \\ &= \frac{1}{\cos^2 x},\end{aligned}$$

so that

$$\frac{d}{dx}(\tan x) = \boxed{\sec^2 x}.$$

This could also be written in other equivalent forms, such as  $1 + \tan^2 x$ .

### Solution to Exercise 3

Since

$$x = \frac{1}{2}at^2,$$

we have

$$\begin{aligned}\frac{dx}{dt} &= at \\ &= \sqrt{2ax},\end{aligned}$$

so that

$$v(t) = \boxed{at},$$

$$v(x) = \boxed{\sqrt{2ax}}.$$

If  $t = 4$ , then  $v = 4a$ , so that

$$v(t = 4) = \boxed{4a}.$$

If  $x = 4$ , then  $v = \sqrt{8a}$ , so that

$$v(x = 4) = \boxed{\sqrt{8a}}.$$

$v(t = x)$  is the expression for  $v(t)$  with  $x$  replacing  $t$ ,

$$v(t = x) = \boxed{ax}.$$

$v(x = t)$  is the expression for  $v(x)$  with  $t$  replacing  $x$ ,

$$v(x = t) = \boxed{\sqrt{2at}}.$$

$v|_{x=3}$  is the same as  $v(x = 3)$ , that is

$$v|_{x=3} = \boxed{\sqrt{6a}}.$$

The derivative of  $v$  with respect to  $t$  is  $d(at)/dt$ , so that

$$\frac{dv}{dt} = \boxed{a}.$$

The derivative of  $v$  with respect to  $x$  is  $d(\sqrt{2ax})/dx$ , which is

$$\sqrt{2a} \frac{d}{dx}(x^{\frac{1}{2}}) = \frac{1}{2} \sqrt{2a} x^{-\frac{1}{2}},$$

or

$$\frac{dv}{dx} = \boxed{\sqrt{\frac{a}{2x}}}.$$

If  $x = 1$ , then  $dv/dx = \sqrt{(a/2)}$ , or

$$\left. \frac{dv}{dx} \right|_{x=1} = \boxed{\sqrt{\frac{a}{2}}}.$$

If  $t = t_0$ , then  $x = \frac{1}{2}at_0^2$ , so that

$$\begin{aligned} \left. \frac{dv}{dx} \right|_{t=t_0} &= \left. \frac{dv}{dx} \right|_{x=\frac{1}{2}at_0^2} \\ &= \sqrt{\frac{a}{2(\frac{1}{2}at_0^2)}} \end{aligned}$$

giving

$$\left. \frac{dv}{dx} \right|_{t=t_0} = \boxed{\frac{1}{t_0}}.$$

## Solution to Exercise 4

From above,

$$\frac{dx}{dt} = \boxed{3t^2}.$$

We have  $v(t) = 3t^2$ ,  $v(x) = 3x^{\frac{2}{3}}$ , so that

$$\frac{dv}{dt} = \boxed{6t},$$

$$\frac{dv}{dx} = \boxed{2x^{-\frac{1}{3}}}.$$

Expressing  $dv/dx$  in terms of  $t$ ,

$$\frac{dv}{dx} = \frac{2}{t},$$

giving

$$\frac{dv}{dx} \frac{dx}{dt} = \boxed{6t}.$$

$$\text{Is } \frac{dv}{dt} = \frac{dv}{dx} \frac{dx}{dt} ? \quad \boxed{\text{Yes}}$$

## Solution to Exercise 5

Since  $x = \frac{1}{2}at^2$  in Exercise 3,

$$x(t) \text{ is } t \longmapsto \boxed{\frac{1}{2}at^2}.$$

Differentiating with respect to  $t$ ,

$$\frac{dx}{dt} = \boxed{at}.$$

From Exercise 3,

$$\frac{dv}{dt} = \boxed{a},$$

and

$$\frac{dv}{dx} = \boxed{\sqrt{\frac{a}{2x}}}.$$

Expressing this in terms of  $t$ ,

$$\frac{dv}{dx} = \frac{1}{t},$$

so that

$$\frac{dv}{dx} \frac{dx}{dt} = \boxed{a}.$$

$$\text{Is } \frac{dv}{dt} = \frac{dv}{dx} \frac{dx}{dt} ? \quad \boxed{\text{Yes}}$$

**Solution to Exercise 6**

If  $z = z(y)$ ,  $y = y(x)$  and  $x = x(t)$ , we know that the chain rule gives both

$$\frac{dz}{dt} = \frac{dz}{dy} \frac{dy}{dt}$$

and

$$\frac{dy}{dt} = \frac{dy}{dx} \frac{dx}{dt}.$$

Combining these two,

$$\frac{dz}{dt} = \boxed{\frac{dz}{dy} \frac{dy}{dx} \frac{dx}{dt}}.$$

Put  $z = \ln y$ ,  $y = \cos x$ ,  $x = e^{2t}$ . Then

$$\frac{dz}{dy} = \frac{1}{y} = \frac{1}{\cos x} = \frac{1}{\cos e^{2t}},$$

$$\frac{dy}{dx} = -\sin x = -\sin e^{2t},$$

$$\frac{dx}{dt} = 2e^{2t},$$

so that

$$\begin{aligned} \frac{dz}{dt} &= \frac{1}{\cos e^{2t}} (-\sin e^{2t}) 2e^{2t} \\ &= -2e^{2t} \tan e^{2t}. \end{aligned}$$

Thus,

$$\frac{d}{dt}(\ln \cos e^{2t}) = \boxed{-2e^{2t} \tan e^{2t}}.$$

**Solution to Exercise 7**

If  $y(x) = \arctan x$ , then

$$x(y) = \boxed{\tan y};$$

hence, by Exercise 2,

$$\frac{dx}{dy}(y) = \boxed{\sec^2 y};$$

hence, by the chain rule,

$$\frac{dy}{dx}(x) = \boxed{\cos^2 y}.$$

Since  $\sec^2 y = 1 + \tan^2 y$ ,

$$\cos^2 y = \frac{1}{1 + x^2};$$

thus

$$\frac{dy}{dx}(x) = \boxed{\frac{1}{1 + x^2}},$$



so that

$$\frac{d}{dx}(\arctan x) = \boxed{\frac{1}{1+x^2}}.$$

## 0S.1.2 Integration

### Solution to Exercise

The methods described below are all based very firmly on the M201 supplementary handbook, *Techniques of Integration (TI)*. If you know a quicker method, so much the better. The constant of integration is omitted in each indefinite integral.

- (i)  $\int x \, dx$ . The integrand  $x$  is a polynomial function of  $x$ , so we can use the formula in section III.1 on page 8 of *TI*, with  $a_1 = 1$ , and  $a_i = 0$  for all  $i \neq 1$ , to get

$$\int x \, dx = \boxed{\frac{1}{2}x^2}.$$

- (ii)  $\int t^2 \, dt$ . The integrand is a polynomial function of  $t$ . We use the same formula as in part (i), but on this occasion  $t$  replaces  $x$  and we have  $a_2 = 1$  and  $a_i = 0$  for all  $i \neq 2$ . Thus

$$\int t^2 \, dt = \boxed{\frac{1}{3}t^3}.$$

- (iii)  $\int e^{-x} \, dx$ . The integrand involves an exponential function, so we look at section III.5 on page 14 of *TI*. Formula 5.2(i) with  $a = -1$ ,  $b = 0$  gives

$$\int e^{-x} \, dx = \boxed{-e^{-x}}.$$

Alternatively formula 5.2(iii) with  $a = -1$ ,  $n = 0$  (or even formula 5.2(vi) with  $a = e^{-1}$ ) gives the same answer.

- (iv)  $\int \frac{1}{1+x^2} \, dx$ . The integrand is a rational function of  $x$ , so we can use section III.2 on page 8 of *TI*. The first relevant paragraph is 2.3, with  $p = 0$ ,  $q = 1$ ,  $b = 0$ ,  $c = 1$  (and note that  $b^2 < 4c$ ; if this had not been so we would have had to go on to paragraph 2.4). The recommended substitution  $t = x + \frac{1}{2}b$  is just  $t = x$ , i.e. we can proceed to the next step immediately. Using the second of the two given standard integrals, with  $c = 1$  and  $x$  in place of  $t$ , we get

$$\int \frac{1}{1+x^2} \, dx = \boxed{\arctan x}.$$

The arctan function was mentioned in Exercise 7 of subsection 0.1.1, and you may have been able to answer this question by using the result of that exercise. A graph of the arctan function is shown on page 7 of *TI*.

- (v)  $\int \frac{1}{4+x^2} \, dx$ . As in part (iv), we are led to paragraph 2.3 in part III of *TI*. This time we have  $p = 0$ ,  $q = 1$ ,  $b = 0$ ,  $c = 4$ , and so the second standard integral at the top of page 9 gives

$$\int \frac{1}{4+x^2} \, dx = \boxed{\frac{1}{2} \arctan \frac{1}{2}x}.$$

- (vi)  $\int \sin \alpha x \sin \beta x \, dx$ . The integrand involves trigonometric functions, so we turn to section III.5 on page 14 of *TI*. Using the first formula in paragraph 5.1, with  $\alpha, \beta$  replacing  $a, b$  respectively, brings the integral to the form

$$\frac{1}{2} \int \cos(\alpha - \beta)x \, dx - \frac{1}{2} \int \cos(\alpha + \beta)x \, dx.$$

The next relevant formula is 5.2(i) with  $a = 0$  and  $b$  put equal to  $\alpha - \beta$  and  $\alpha + \beta$  in turn. This gives

$$\int \sin \alpha x \sin \beta x \, dx = \boxed{\frac{1}{2} \left[ \frac{\sin(\alpha - \beta)x}{\alpha - \beta} - \frac{\sin(\alpha + \beta)x}{\alpha + \beta} \right]}.$$

- (vii)  $\int_0^\pi \sin mx \sin nx \, dx$ . We have two cases to consider, depending upon whether or not  $m$  and  $n$  are equal. Suppose  $m \neq n$ . We then have from the answer to part (vi), with  $m, n$  replacing  $\alpha, \beta$  respectively,

$$\begin{aligned} \int_0^\pi \sin mx \sin nx \, dx &= \frac{1}{2} \left[ \frac{\sin(m - n)x}{m - n} - \frac{\sin(m + n)x}{m + n} \right]_0^\pi \\ &= 0, \end{aligned}$$

since the sine of any integer multiple of  $\pi$  is zero. Suppose next that  $m = n$ . Our integrand is now  $\sin^2 nx$ , and using the first formula in paragraph 5.1 (section III.5 on page 14 of *TI*) with  $a = b = n$ ,

$$\begin{aligned} \int_0^\pi \sin^2 nx \, dx &= \frac{1}{2} \int_0^\pi (1 - \cos 2nx) \, dx \\ &= \frac{1}{2} \left[ x - \frac{1}{2n} \sin 2nx \right]_0^\pi, \end{aligned}$$

having used formula 5.2(i) with  $a = 0, b = 2n$  to integrate the second term of the integrand. Thus

$$\int_0^\pi \sin^2 nx \, dx = \frac{1}{2}\pi,$$

so that

$$\int_0^\pi \sin mx \sin nx \, dx = \begin{cases} \boxed{\frac{1}{2}\pi} & \text{if } m = n, \\ \boxed{0} & \text{if } m \neq n. \end{cases}$$

- (viii)  $\int_0^\pi \cos x \sin nx \, dx$ . We again start with paragraph 5.1 of section III.5 on page 14 of *TI*. Using the third formula in this paragraph, with  $a = n, b = 1$ , our integral becomes

$$\frac{1}{2} \int_0^\pi \sin(n - 1)x \, dx + \frac{1}{2} \int_0^\pi \sin(n + 1)x \, dx.$$

We then use formula 5.2(ii) with  $a = 0$  and  $b$  equal to  $n - 1, n + 1$  in turn to get

$$\int_0^\pi \cos x \sin nx \, dx = \frac{1}{2} \left[ -\frac{\cos(n - 1)x}{n - 1} - \frac{\cos(n + 1)x}{n + 1} \right]_0^\pi$$

provided that  $n \neq 1$  (if  $n = 1$ , we see that the first integrand above is zero, so that the first term in the square bracket should be replaced by zero in this case). Since  $\cos m\pi = (-1)^m$  if  $m$  is an integer,

$$\int_0^\pi \cos x \sin nx \, dx = \begin{cases} \boxed{\frac{1}{n - 1} + \frac{1}{n + 1}} & \text{if } n \text{ is even,} \\ \boxed{0} & \text{if } n \text{ is odd.} \end{cases}$$

- (ix)  $\int_0^\pi x \sin nx \, dx$ . We can perform the integration by using formula 5.2(v) of section III.5 in *TI* with  $n$  replaced by 1 and  $a$  replaced by  $n$ . This gives

$$\begin{aligned}\int_0^\pi x \sin nx \, dx &= \left[ -\frac{x}{n} \cos nx + \frac{1}{n^2} \sin nx \right]_0^\pi \\ &= -\frac{\pi}{n} \cos n\pi,\end{aligned}$$

so that

$$\int_0^\pi x \sin nx \, dx = \boxed{(-1)^{n+1} \frac{\pi}{n}}.$$

- (x)  $\int_{-\pi}^\pi x^2 \cos nx \, dx$ . Using formula 5.2(iv) of section III.5 with  $n$  replaced by 2 and  $a$  replaced by  $n$ ,

$$\begin{aligned}\int_{-\pi}^\pi x^2 \cos nx \, dx &= \left[ \left( \frac{x^2}{n} - \frac{2}{n^3} \right) \sin nx + \frac{2x}{n^2} \cos nx \right]_{-\pi}^\pi \\ &= \frac{2\pi}{n^2} \cos n\pi - \frac{2(-\pi)}{n^2} \cos(-n\pi) \\ &= \frac{4\pi}{n^2} \cos n\pi,\end{aligned}$$

so that

$$\int_{-\pi}^\pi x^2 \cos nx \, dx = \boxed{4(-1)^n \frac{\pi}{n^2}}.$$

- (xi)  $\int_{-\pi}^\pi f(x) \sin nx \, dx$ . Since  $f(x) = 0$  for  $-\pi \leq x \leq 0$  and  $f(x) = 1$  for  $0 < x \leq \pi$ , we break our integral up into two parts:

$$\begin{aligned}\int_{-\pi}^\pi f(x) \sin nx \, dx &= \int_{-\pi}^0 0 \times \sin nx \, dx + \int_0^\pi 1 \times \sin nx \, dx \\ &= \int_0^\pi \sin nx \, dx \\ &= \left[ -\frac{1}{n} \cos nx \right]_0^\pi,\end{aligned}$$

having used formula 5.2(ii) with  $a = 0$ ,  $b = n$ . Since

$$\left[ -\frac{1}{n} \cos nx \right]_0^\pi = -\frac{1}{n} \cos n\pi + \frac{1}{n},$$

it follows that

$$\int_{-\pi}^\pi f(x) \sin nx \, dx = \boxed{\frac{1}{n}(1 - (-1)^n)}.$$

- (xii)  $\int_0^1 f(x) \sin n\pi x \, dx$ . Since  $f(x) = x$  for  $0 \leq x \leq \frac{1}{2}$  and  $f(x) = 1 - x$  for  $\frac{1}{2} \leq x \leq 1$ , we break our integral into two parts:

$$\begin{aligned}\int_0^1 f(x) \sin n\pi x \, dx &= \int_0^{\frac{1}{2}} f(x) \sin n\pi x \, dx + \int_{\frac{1}{2}}^1 f(x) \sin n\pi x \, dx \\ &= \int_0^{\frac{1}{2}} x \sin n\pi x \, dx + \int_{\frac{1}{2}}^1 (1 - x) \sin n\pi x \, dx.\end{aligned}$$

We see by putting  $n = 1$ ,  $a = n\pi$  in formula 5.2(v) in section III.5 on page 14 of *TI* that

$$\int x \sin n\pi x \, dx = -\frac{x}{n\pi} \cos n\pi x + \frac{1}{n^2\pi^2} \sin n\pi x.$$

By putting  $n = 0$ ,  $a = n\pi$  in the same formula,

$$\int \sin n\pi x \, dx = -\frac{1}{n\pi} \cos n\pi x.$$

Thus,

$$\begin{aligned} \int_0^1 f(x) \sin n\pi x \, dx &= \left[ -\frac{x}{n\pi} \cos n\pi x + \frac{1}{n^2\pi^2} \sin n\pi x \right]_0^{\frac{1}{2}} \\ &\quad + \left[ -\frac{1}{n\pi} \cos n\pi x + \frac{x}{n\pi} \cos n\pi x - \frac{1}{n^2\pi^2} \sin n\pi x \right]_{\frac{1}{2}}^1 \\ &= -\frac{1}{2n\pi} \cos \frac{1}{2}n\pi + \frac{1}{n^2\pi^2} \sin \frac{1}{2}n\pi \\ &\quad + \frac{1}{2n\pi} \cos \frac{1}{2}n\pi + \frac{1}{n^2\pi^2} \sin \frac{1}{2}n\pi \\ &= \frac{2}{n^2\pi^2} \sin \frac{1}{2}n\pi. \end{aligned}$$

Now  $\sin \frac{1}{2}n\pi$  is zero if  $n$  is even. If  $n$  is odd,  $\sin \frac{1}{2}n\pi = 1$  if  $\frac{1}{2}(n-1)$  is even and  $\sin \frac{1}{2}n\pi = -1$  if  $\frac{1}{2}(n-1)$  is odd.

Therefore,

$$\int_0^1 f(x) \sin n\pi x \, dx = \begin{cases} 0 & \text{if } n \text{ is even,} \\ (-1)^{\frac{1}{2}(n-1)} \frac{2}{n^2\pi^2} & \text{if } n \text{ is odd.} \end{cases}$$

### 0S.1.3 Differential Equations of First Order and Degree

#### Solution to Exercise

- (i) Integrating both sides we obtain

$$x(t) = \frac{1}{3}t^3 + A.$$

- (ii) Separating variables (see page 20 of *Unit M100 24* or section 4.2 of *SS* if necessary),

$$\int e^{-x} \, dx = \int dt + A,$$

or

$$-e^{-x} = t + A.$$

Therefore

$$x(t) = -\ln(-t - A) \quad t \in (-\infty, -A).$$

Notice that in this case no solution has as domain the whole of  $R$ . This commonly happens with nonlinear differential equations, for which the simple uniqueness theorem studied in Units M201 4 and M201 33 may not apply.

- (iii) Separating variables,

$$\int x \, dx = \int t \, dt + C,$$

giving

$$\frac{1}{2}x^2 = \frac{1}{2}t^2 + C,$$

so that

$$x(t) = \boxed{\pm \sqrt{t^2 + A}},$$

where  $A = 2C$ , and  $t \in R$  if  $A \geq 0$ ,  $t \in (\sqrt{-A}, \infty) \cup (-\infty, -\sqrt{-A})$  if  $A < 0$ .

- (iv) Separating variables,

$$\int \frac{1}{1+x^2} dx = \int dt + A,$$

and using part (iv) of the Exercise in subsection 0.1.2 this gives

$$\arctan x = t + A,$$

or

$$x(t) = \boxed{\tan(t + A)} \quad t \in (-A - \frac{1}{2}\pi, -A + \frac{1}{2}\pi).$$

- (v) Separating variables,

$$\int \frac{1}{4+x^2} dx = \int dt + C,$$

so that using part (v) of the Exercise in subsection 0.1.2 gives

$$\frac{1}{2}\arctan \frac{1}{2}x = t + C,$$

or

$$x(t) = \boxed{2 \tan(2t + A)} \quad t \in (-\frac{1}{2}A - \frac{1}{4}\pi, -\frac{1}{2}A + \frac{1}{4}\pi),$$

where  $A = 2C$ .

- (vi) The equation can be written as

$$\frac{dx}{dt} - \frac{1}{t}x = 0,$$

so that an integrating factor is

$$\exp \int \left(-\frac{1}{t}\right) dt = \exp(-\ln t) = \frac{1}{t}$$

(see lines 6 to 10 on page 97 of  $K$  if necessary). Multiplying through by this factor, we see that the equation can be put in the form

$$\frac{d}{dt} \left( \frac{x}{t} \right) = 0,$$

which has solution

$$x(t) = \boxed{At}.$$

### 0S.1.4 Differential Equations of First Order and Second Degree

#### Solution to Exercise

- (i) Factorizing the equation gives

$$\left(2\frac{dx}{dt} - 1\right)\left(\frac{dx}{dt} + 1\right) = 0,$$

so that

$$\frac{dx}{dt} = \frac{1}{2} \text{ or } -1.$$

(Alternatively you can use the general quadratic equation formula

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

to achieve this result.)

Integrating,

$$x(t) = \boxed{-t + A} \quad \text{or} \quad \boxed{\frac{1}{2}t + A}.$$

- (ii) Factorizing gives

$$\left(\frac{dx}{dt} - e^x\right)\left(\frac{dx}{dt} - t^2\right) = 0,$$

so that

$$\frac{dx}{dt} = e^x \quad \text{or} \quad t^2.$$

Hence from parts (i) and (ii) of the Exercise in subsection 0.1.3,

$$x(t) = \boxed{-\ln(-t - A)} \quad \text{or} \quad \boxed{\frac{1}{3}t^3 + A},$$

with the restriction that  $t \in (-\infty, -A)$  in the first case.

## 0S.2 FIRST-ORDER PARTIAL DIFFERENTIATION

### 0S.2.1 First-Order Partial Derivatives

#### Solution to Exercise 1

$$\begin{aligned}
 \text{(i)} \quad \frac{\partial u}{\partial x}(x, y) &= \boxed{2x} \\
 \frac{\partial u}{\partial y}(x, y) &= \boxed{2y} \\
 \text{(ii)} \quad \frac{\partial u}{\partial x}(x, y) &= \boxed{y^2 e^{xy}} \\
 \frac{\partial u}{\partial y}(x, y) &= \boxed{(1 + xy)e^{xy}} \\
 \text{(iii)} \quad \frac{\partial u}{\partial x}(x, y) &= \boxed{\cos(x - y)} \\
 \frac{\partial u}{\partial y}(x, y) &= \boxed{-\cos(x - y)} \\
 \text{(iv)} \quad \frac{\partial u}{\partial x}(x, y) &= \boxed{f'(x - y)} \\
 \frac{\partial u}{\partial y}(x, y) &= \boxed{-f'(x - y)}
 \end{aligned}$$

#### Solution to Exercise 2

$$\begin{aligned}
 \frac{\partial}{\partial x}[p(x + ct) + q(x - ct)] &= \boxed{p'(x + ct) + q'(x - ct)} \\
 \frac{\partial}{\partial t}[p(x + ct) + q(x - ct)] &= \boxed{cp'(x + ct) - cq'(x - ct)}
 \end{aligned}$$

#### Solution to Exercise 3

- (i) Suppose that  $G$  is a primitive of  $g$ , so that by the fundamental theorem of calculus,  $G' = g$ . Then

$$\begin{aligned}
 \int_0^{x+ct} g(\bar{x}) d\bar{x} &= \left[ G(\bar{x}) \right]_0^{x+ct} \\
 &= G(x + ct) - G(0),
 \end{aligned}$$

so that

$$\begin{aligned}
 \frac{\partial}{\partial x} \int_0^{x+ct} g(\bar{x}) d\bar{x} &= \frac{\partial}{\partial x} [G(x + ct) - G(0)] \\
 &= G'(x + ct),
 \end{aligned}$$

or

$$\frac{\partial}{\partial x} \int_0^{x+ct} g(\bar{x}) d\bar{x} = \boxed{g(x + ct)}$$

- (ii) This is the same as for part (i), with  $x - ct$  replacing  $x + ct$  throughout. Thus

$$\frac{\partial}{\partial x} \int_0^{x-ct} g(\bar{x}) d\bar{x} = \boxed{g(x - ct)}$$

(iii) Since

$$\int_{x-ct}^{x+ct} g(\bar{x}) d\bar{x} = \int_0^{x+ct} g(\bar{x}) d\bar{x} - \int_0^{x-ct} g(\bar{x}) d\bar{x},$$

we have from parts (i) and (ii)

$$\frac{\partial}{\partial x} \int_{x-ct}^{x+ct} g(\bar{x}) d\bar{x} = \boxed{g(x+ct) - g(x-ct)}.$$

(iv) Defining  $G$  as before,

$$\frac{\partial}{\partial t} \int_{x-ct}^{x+ct} g(\bar{x}) d\bar{x} = \frac{\partial}{\partial t} [G(x+ct) - G(x-ct)].$$

Using the second result of Exercise 2 with  $p = G$ ,  $q = -G$ ,

$$\frac{\partial}{\partial t} \int_{x-ct}^{x+ct} g(\bar{x}) d\bar{x} = \boxed{cg(x+ct) + cg(x-ct)}.$$

#### Solution to Exercise 4

If  $u(x, t) = \sin(x^2 - t)$ , put  $t = 0$  to get  $u(x, 0) = u|_{t=0}$ :

$$u|_{t=0} = \boxed{\sin x^2}.$$

The partial derivative of  $u$  with respect to  $t$  is

$$\frac{\partial u}{\partial t}(x, t) = -\cos(x^2 - t),$$

so that

$$\left. \frac{\partial u}{\partial t} \right|_{t=0} = \boxed{-\cos x^2}.$$

The partial derivative of  $u$  with respect to  $x$  is

$$\frac{\partial u}{\partial x}(x, t) = \boxed{2x \cos(x^2 - t)}.$$

Also,

$$u|_{x=0} = \boxed{-\sin t},$$

$$\left. \frac{\partial u}{\partial x} \right|_{x=1} = \boxed{2 \cos(1 - t)}.$$



## 0S.2.2 Integration of Partial Derivatives

### Solution to Exercise 1

- (i) Since partial differentiation of a function of  $x$  and  $y$  with respect to  $x$  is performed by treating  $y$  as a constant,

$$\frac{\partial u}{\partial x} = \boxed{0}.$$

- (ii) As in part (i),

$$\frac{\partial u}{\partial x} = \boxed{0}.$$

- (iii) As in parts (i) and (ii),

$$\frac{\partial u}{\partial x} = \boxed{0}.$$

### Solution to Exercise 2

The general solution is the function we differentiated in Exercise 1(iii):

$$u(x, y) = \boxed{\begin{array}{l} f(y), \\ \text{where } f \text{ is an arbitrary function} \\ \text{of one variable.} \end{array}}$$

### Solution to Exercise 3

The general solution is

$$u(x, y) = \boxed{\begin{array}{l} f(x), \\ \text{where } f \text{ is an arbitrary function} \\ \text{of one variable.} \end{array}}$$

A brief justification:

For fixed  $x$ , consider

$$y \mapsto u(x, y).$$

This has derived function

$$y \mapsto \frac{\partial u}{\partial y}(x, y),$$

which is

$$y \mapsto 0$$

by the differential equation. Hence  $y \mapsto u(x, y)$  is a constant function (regarding  $x$  as a fixed number) so that

$$\begin{aligned} u(x, y) &= u(x, 0) \\ &= f(x) \quad (\text{say}). \end{aligned}$$

**Solution to Exercise 4**

- (i) The general solution of the associated homogeneous equation is

$$u(x, y) = f(y)$$

for arbitrary  $f$ . If we regard  $y$  as fixed, a particular primitive of the function

$$x \mapsto y$$

is

$$x \mapsto xy,$$

so that the general solution we require is

$$u(x, y) = \boxed{xy + f(y)}.$$

- (ii) Since for fixed
- $x$
- the function

$$y \mapsto xy$$

has a primitive

$$y \mapsto \frac{1}{2}xy^2,$$

the general solution is

$$u(x, y) = \boxed{\frac{1}{2}xy^2 + f(x)}.$$

- (iii) Since for fixed
- $t$
- the function

$$x \mapsto x^2 + t^2$$

has a primitive

$$x \mapsto \frac{1}{3}x^3 + xt^2,$$

the general solution is

$$u(x, t) = \boxed{\frac{1}{3}x^3 + xt^2 + f(t)}.$$

- (iv) Since for fixed
- $\xi$
- the function

$$\eta \mapsto r(\eta)$$

has a primitive

$$\eta \mapsto \int r(\eta) d\eta,$$

the general solution is

$$w(\xi, \eta) = \boxed{\int r(\eta) d\eta + f(\xi)}.$$

**Solution to Exercise 5**

- (i) By the standard formula (see lines 6 to 10 on page 97 of
- $K$
- ), an integrating factor for the ordinary differential equation is

$$\exp \int \left( -\frac{1}{x} \right) dx = \exp(-\ln x),$$

so that the integrating factor is

$$\boxed{\frac{1}{x}}.$$

(ii) Using this factor for the partial differential equation we obtain

$$\frac{1}{x} \frac{\partial u}{\partial x} - \frac{u}{x^2} = 0,$$

or

$$\frac{\partial}{\partial x} \left( \frac{u(x, y)}{x} \right) = 0.$$

Thus

$$\frac{u(x, y)}{x} = f(y),$$

and the general solution is

$$u(x, y) = \boxed{xf(y)}.$$

### Solution to Exercise 6

(i) The integrating factor is

$$\exp \int \left( -\frac{1}{y} \right) dx = \boxed{\exp \left( -\frac{x}{y} \right)}.$$

(ii) The partial differential equation becomes

$$\frac{\partial}{\partial x} \left( u(x, y) \exp \left( -\frac{x}{y} \right) \right) = 0,$$

so that the general solution is

$$u(x, y) = \boxed{e^{x/y} f(y)}.$$

## 0S.2.3 Partial Differential Operators

### Solution to Exercise 1

Since the operator is to be first-order, we are limited to first- and zeroth-order derivatives, and there can be no products of the first-order derivatives since this would contravene the linearity of  $L$ . Thus the most general form possible for  $L$  is

$$L = \boxed{a(x, y) \frac{\partial}{\partial x} + b(x, y) \frac{\partial}{\partial y} + c(x, y)},$$

for arbitrary functions of two variables  $a$ ,  $b$  and  $c$ .

### Solution to Exercise 2

(i) Since

$$\frac{\partial}{\partial x} (x + y)^2 = \frac{\partial}{\partial y} (x + y)^2 = 2(x + y),$$

it follows that

$$\left( \frac{\partial}{\partial x} + \frac{\partial}{\partial y} \right) (x + y)^2 = \boxed{4(x + y)}.$$

(ii) Since

$$\frac{\partial}{\partial x}(x - y)^2 = -\frac{\partial}{\partial y}(x - y)^2 = 2(x - y),$$

we have

$$\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right)(x - y)^2 = \boxed{0}.$$

(iii) Since

$$\frac{\partial}{\partial x}\sin(x - y) = -\frac{\partial}{\partial y}\sin(x - y) = \cos(x - y),$$

we have

$$\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right)\sin(x - y) = \boxed{0}.$$

(iv) Since

$$\frac{\partial}{\partial x}F(x, y) = -\frac{\partial}{\partial y}F(x - y) = F'(x - y),$$

we have

$$\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right)F(x - y) = \boxed{0}.$$

## 0S.2.4 Differentiation Along a Curve: the Chain Rule

### Solution to Exercise 1

(i) Put

$$x = 2t, y = \sin t$$

into the potential energy  $\phi = x^2 + y^3$  to get

$$\phi(t) = \boxed{4t^2 + \sin^3 t}.$$

(ii) Differentiating this expression for  $\phi(t)$  gives

$$\phi'(t) = \frac{d\phi}{dt} = \boxed{8t + 3 \sin^2 t \cos t}.$$

### Solution to Exercise 2

(i) We have

$$z = x^2 + y^3,$$

and

$$x = \xi(t), y = \eta(t),$$

$$\text{so that } z(t) = \boxed{[\xi(t)]^2 + [\eta(t)]^3}.$$

(ii) Differentiating with respect to  $t$  gives

$$z'(t) = \frac{dz}{dt} = \boxed{2\xi(t)\xi'(t) + 3[\eta(t)]^2\eta'(t)}$$

**Solution to Exercise 3**

(i) Since  $x = \xi(t)$ , we have

$$\frac{dx}{dt} = \boxed{\xi'(t)}.$$

(ii) If  $z = x^2 + y^3$ , we have

$$\frac{\partial z}{\partial x} = \boxed{2x}.$$

(iii) We want  $\partial z/\partial x$  evaluated at  $x = \xi(t)$ ,  $y = \eta(t)$ . From part (ii) we have

$$\frac{\partial z}{\partial x}(\xi(t), \eta(t)) = \boxed{2\xi(t)}.$$

(iv) By proceeding as in parts (i), (ii) and (iii) we find that

$$\begin{aligned} \frac{dy}{dt} &= \eta'(t), \\ \frac{\partial z}{\partial y}(\xi(t), \eta(t)) &= 3[\eta(t)]^2, \end{aligned}$$

so that at  $(\xi(t), \eta(t))$  on the surface,

$$\frac{\partial z}{\partial x} \frac{dx}{dt} + \frac{\partial z}{\partial y} \frac{dy}{dt} = \boxed{2\xi(t)\xi'(t) + 3[\eta(t)]^2\eta'(t)}.$$

This is the same as the formula for  $dz/dt$  in Exercise 2(ii).

**Solution to Exercise 4**

(i) This is the case in Exercises 2 and 3 (if  $z$  is replaced by  $u$ ). Since the solutions to Exercises 2(ii) and 3(iv) are identical, we see that the boxed formula holds in this case.

Yes

(ii) The result is also true in general; a demonstration is given in Appendix I of *Unit MST 282 7* on pages 34, 35.

Yes

**Solution to Exercise 5**

The analogue of the formula in Exercise 4 for the case where  $u$  is a function of three variables  $x, y, z$  is

$$\boxed{\frac{du}{dt} = \frac{\partial u}{\partial x} \frac{dx}{dt} + \frac{\partial u}{\partial y} \frac{dy}{dt} + \frac{\partial u}{\partial z} \frac{dz}{dt}}$$

### Solution to Exercise 6

(i) We have

$$\frac{\partial u}{\partial x} = 2x = 2t^2, \quad \frac{dx}{dt} = 2t,$$

$$\frac{\partial u}{\partial y} = 2y = 2(t-1) \quad \frac{dy}{dt} = 1,$$

$$\frac{\partial u}{\partial z} = 2z = 2t^3, \quad \frac{dz}{dt} = 3t^2,$$

so that

$$\frac{\partial u}{\partial x} \frac{dx}{dt} + \frac{\partial u}{\partial y} \frac{dy}{dt} + \frac{\partial u}{\partial z} \frac{dz}{dt} = (2t^2)(2t) + 2(t-1) + (2t^3)(3t^2),$$

giving

$$\frac{du}{dt} = \boxed{6t^5 + 4t^3 + 2t - 2}.$$

(ii) Writing

$$u = (t^2)^2 + (t-1)^2 + (t^3)^2$$

or

$$u(t) = \boxed{t^6 + t^4 + t^2 - 2t + 1},$$

we have

$$\frac{du}{dt} = \boxed{6t^5 + 4t^3 + 2t - 2}.$$

The two methods agree.

### Solution to Exercise 7

(i) By definition we have

$$\text{grad } u = \frac{\partial u}{\partial x} \mathbf{i} + \frac{\partial u}{\partial y} \mathbf{j} + \frac{\partial u}{\partial z} \mathbf{k},$$

so that by reference to the solution of Exercise 6,

$$\text{grad } u = \boxed{2(t^2 \mathbf{i} + (t-1) \mathbf{j} + t^3 \mathbf{k})}.$$

(ii) If  $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ , then

$$\frac{d\mathbf{r}}{dt} = \frac{dx}{dt} \mathbf{i} + \frac{dy}{dt} \mathbf{j} + \frac{dz}{dt} \mathbf{k},$$

or

$$\frac{d\mathbf{r}}{dt} = \boxed{2t \mathbf{i} + \mathbf{j} + 3t^2 \mathbf{k}}.$$

$$\begin{aligned} \text{(iii)} \quad \text{grad } u \cdot \frac{d\mathbf{r}}{dt} &= 2(t^2 \mathbf{i} + (t-1) \mathbf{j} + t^3 \mathbf{k}) \cdot (2t \mathbf{i} + \mathbf{j} + 3t^2 \mathbf{k}) \\ &= 2(2t^3 + (t-1) + 3t^5), \end{aligned}$$

or

$$\text{grad } u \cdot \frac{d\mathbf{r}}{dt} = \boxed{6t^5 + 4t^3 + 2t - 2},$$

which is the same as the expression for  $du/dt$  derived in Exercise 6.

### 0S.2.5 Derivatives “Following the Motion”

#### Solution to Exercise 1

The velocity at any point  $x$  in the centre of the stream at time  $t$  is  $v(x, t)$ . If the motion of the leaf (which is the same as that of the water immediately beneath it) is given by  $x = x(t)$ , its velocity at time  $t$  will be  $v(x(t), t)$ . But we can also find this velocity by differentiating the function  $x(t)$ . Thus

$$\frac{dx}{dt} = v(x(t), t).$$

#### Solution to Exercise 2

The temperature  $T(t)$  at the leaf at time  $t$  will be equal to the temperature  $T(x, t)$  at position  $x$  and time  $t$ , where  $x = x(t)$ . That is,

$$T(t) = T(x(t), t).$$

#### Solution to Exercise 3

(i) Since

$$T(x, t) = b \ln x + c \sin t,$$

and

$$x(t) = at + k,$$

we have

$$T(x(t), t) = b \ln(at + k) + c \sin t.$$

Thus, from the solution to Exercise 2,

$$T(t) = b \ln(at + k) + c \sin t.$$

(ii) Differentiating this expression,

$$\frac{dT}{dt} = \frac{ab}{at + k} + c \cos t.$$

(The first term of  $T(t)$  can be differentiated using the chain rule for functions of one variable,

$$\frac{d}{dt}(b \ln x) = \frac{d}{dx}(b \ln x) \frac{dx}{dt},$$

where  $x = at + k$ .)

**Solution to Exercise 4**

Consider the data of Exercise 3. Computing the quantities to be included in the general expression for  $dT/dt$  we see that in Exercise 3 these were

$$\frac{\partial T}{\partial x} = \frac{b}{x} = \frac{b}{at + k},$$

$$\frac{\partial T}{\partial t} = c \cos t,$$

and

$$\frac{dx}{dt} = a.$$

Comparing with the solution to Exercise 3, we see that we can write

$$\frac{dT}{dt} = \boxed{\frac{dx}{dt} \frac{\partial T}{\partial x} + \frac{\partial T}{\partial t}}.$$

We could alternatively derive this result from the (boxed) chain rule in Exercise 4 of subsection 0.2.4 upon replacing  $u$  by  $T$  and  $y$  by  $t$ .

**Solution to Exercise 5**

The text of the exercise after the first sentence should read as follows:

"The symbol  $T$  on the left refers to the function  $\boxed{T(t)}$  with domain  $\boxed{R}$ , describing the temperature at the leaf at time  $t$ . The symbol  $T$  on the right refers to the function  $\boxed{T(x, t)}$ , with domain  $\boxed{R \times R}$ , describing the temperature at position  $\boxed{x}$  at time  $t$ ."

**Solution to Exercise 6**

Replacing  $T$  by  $v$  in the solution to Exercise 4, we find

$$\frac{dv}{dt} = \boxed{\frac{dx}{dt} \frac{\partial v}{\partial x} + \frac{\partial v}{\partial t}}.$$

**Solution to Exercise 7**

The operator we seek is  $d/dt$  (as it appears on the left-hand sides of the solutions to Exercises 4 and 6). If we remove the particular dependent variable ( $v$  or  $T$ ) involved in either of those expressions, we are left with

$$\frac{d}{dt} = \boxed{\frac{dx}{dt} \frac{\partial}{\partial x} + \frac{\partial}{\partial t}}.$$

It was shown in Exercise 1 that  $dx/dt = v(x(t), t)$ . Thus for the particular leaf, the operator can be written as

$$v(x(t), t) \frac{\partial}{\partial x} + \frac{\partial}{\partial t}.$$

We abandon the particular spot occupied by the leaf in favour of an arbitrary point of the stream by replacing the specific function  $x(t)$  by the coordinate  $x$ . Writing  $v$  for  $v(x, t)$ , we are left with the operator equation

$$\frac{d}{dt} = \boxed{v \frac{\partial}{\partial x} + \frac{\partial}{\partial t}}.$$



## 0S.2.6 Change of Coordinates

### Solution to Exercise 1

The expressions for  $x$  and  $y$  are

$$\begin{aligned} x &= \boxed{r \cos \theta}, \\ y &= \boxed{r \sin \theta}. \end{aligned}$$

### Solution to Exercise 2

Since  $T = f(x, y)$ , we can use the formulae from Exercise 1 to get

$$T = \boxed{f(r \cos \theta, r \sin \theta)}.$$

### Solution to Exercise 3

Along any curve for which  $\theta = \text{constant}$  we have  $T = f(x, y)$ , where  $x$  and  $y$  are functions of  $r$  alone. Using the boxed formula

$$\boxed{\frac{du}{dt} = \frac{\partial u}{\partial x} \frac{dx}{dt} + \frac{\partial u}{\partial y} \frac{dy}{dt}}$$

(from Exercise 4 of subsection 0.2.4) with  $T$  for  $u$  and  $r$  for  $t$ , we have

$$\frac{dT}{dr} = \frac{\partial T}{\partial x} \frac{dx}{dr} + \frac{\partial T}{\partial y} \frac{dy}{dr} \text{ along } \theta = \text{constant}.$$

For functions of two variables  $r$  and  $\theta$  however, the operator “ $d/dr$  along  $\theta = \text{constant}$ ” is, by definition, the same as  $\partial/\partial r$ . We can therefore write the above as

$$\frac{\partial T}{\partial r} = \frac{\partial T}{\partial x} \frac{\partial x}{\partial r} + \frac{\partial T}{\partial y} \frac{\partial y}{\partial r}.$$

Since in this case

$$\frac{\partial x}{\partial r} = \cos \theta,$$

$$\frac{\partial y}{\partial r} = \sin \theta,$$

we have

$$\frac{\partial T}{\partial r} = \boxed{\cos \theta \frac{\partial T}{\partial x} + \sin \theta \frac{\partial T}{\partial y}}.$$

Similarly we can write

$$\frac{dT}{d\theta} = \frac{\partial T}{\partial x} \frac{dx}{d\theta} + \frac{\partial T}{\partial y} \frac{dy}{d\theta} \text{ along } r = \text{constant},$$

or

$$\frac{\partial T}{\partial \theta} = \frac{\partial T}{\partial x} \frac{\partial x}{\partial \theta} + \frac{\partial T}{\partial y} \frac{\partial y}{\partial \theta}.$$

Since

$$\frac{\partial x}{\partial \theta} = -r \sin \theta,$$

$$\frac{\partial y}{\partial \theta} = r \cos \theta,$$

we arrive at

$$\frac{\partial T}{\partial \theta} = \boxed{-r \sin \theta \frac{\partial T}{\partial x} + r \cos \theta \frac{\partial T}{\partial y}}.$$

**Solution to Exercise 4**

Since  $T = f(x, y)$ , and  $x = \alpha p + \beta q$ ,  $y = \gamma p + \delta q$ ,

$$T = \boxed{f(\alpha p + \beta q, \gamma p + \delta q)}.$$

We obtain the expressions for  $\partial T/\partial p$  and  $\partial T/\partial q$  as before by using the boxed formula

$$\boxed{\frac{du}{dt} = \frac{\partial u}{\partial x} \frac{dx}{dt} + \frac{\partial u}{\partial y} \frac{dy}{dt}}$$

(from Exercise 4 of subsection 0.2.4) with the total derivatives replaced by partial derivatives,  $T$  for  $u$ , and first  $p$  and then  $q$  in place of  $t$ :

$$\frac{\partial T}{\partial p} = \frac{\partial T}{\partial x} \frac{\partial x}{\partial p} + \frac{\partial T}{\partial y} \frac{\partial y}{\partial p},$$

$$\frac{\partial T}{\partial q} = \frac{\partial T}{\partial x} \frac{\partial x}{\partial q} + \frac{\partial T}{\partial y} \frac{\partial y}{\partial q}.$$

Since

$$\frac{\partial x}{\partial p} = \alpha, \frac{\partial y}{\partial p} = \gamma, \frac{\partial x}{\partial q} = \beta, \frac{\partial y}{\partial q} = \delta,$$

we have

$$\frac{\partial T}{\partial p} = \boxed{\alpha \frac{\partial T}{\partial x} + \gamma \frac{\partial T}{\partial y}},$$

$$\frac{\partial T}{\partial q} = \boxed{\beta \frac{\partial T}{\partial x} + \delta \frac{\partial T}{\partial y}}.$$

**Solution to Exercise 5**

We proceed as before. Since  $T = f(x, y)$  and  $x = x(\xi, \eta)$ ,  $y = y(\xi, \eta)$ ,

$$T = \boxed{f(x(\xi, \eta), y(\xi, \eta))}.$$

The chain rule is used as in Exercises 3 and 4 to give

$$\frac{\partial T}{\partial \xi} = \boxed{\frac{\partial T}{\partial x} \frac{\partial x}{\partial \xi} + \frac{\partial T}{\partial y} \frac{\partial y}{\partial \xi}},$$

$$\frac{\partial T}{\partial \eta} = \boxed{\frac{\partial T}{\partial x} \frac{\partial x}{\partial \eta} + \frac{\partial T}{\partial y} \frac{\partial y}{\partial \eta}}.$$

**Solution to Exercise 6**

The symbols used here are differently arranged from those in the general case of Exercise 5. We are given  $\xi = \xi(x, t)$ ,  $\eta = \eta(x, t)$  (for particular functions  $\xi$  and  $\eta$ ), and the corresponding chain rule is

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial \xi} \frac{\partial \xi}{\partial x} + \frac{\partial u}{\partial \eta} \frac{\partial \eta}{\partial x},$$

$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial \xi} \frac{\partial \xi}{\partial t} + \frac{\partial u}{\partial \eta} \frac{\partial \eta}{\partial t}.$$

Since

$$\xi(x, t) = x + ct,$$

$$\eta(x, t) = x - ct,$$

we have

$$\frac{\partial \xi}{\partial x} = \frac{\partial \eta}{\partial x} = 1,$$

$$\frac{\partial \xi}{\partial t} = -\frac{\partial \eta}{\partial t} = c,$$

so that

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial \xi} + \frac{\partial u}{\partial \eta},$$

$$\frac{\partial u}{\partial t} = c \left( \frac{\partial u}{\partial \xi} - \frac{\partial u}{\partial \eta} \right).$$

**Solution to Exercise 7**

We have  $T(x, z) = T(x, y) = f(x, y)$  where  $y = x + z$ . Using the general form of the chain rule appearing in the last exercise, with  $u, t, \xi, \eta$  being replaced by  $T(x, z) = f(x, y)$ ,  $z, x$  and  $y$  respectively,

$$\frac{\partial T}{\partial x}(x, z) = \frac{\partial f}{\partial x} + \frac{\partial y}{\partial x} \frac{\partial f}{\partial y},$$

$$\frac{\partial T}{\partial z}(x, z) = \frac{\partial y}{\partial z} \frac{\partial f}{\partial y}$$

(since  $\partial x / \partial x = 1$  and  $\partial x / \partial z = 0$ ). Thus

$$\frac{\partial T}{\partial x}(x, z) = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y},$$

$$\frac{\partial T}{\partial z}(x, z) = \frac{\partial f}{\partial y},$$

or

$$\frac{\partial T}{\partial x}(x, z) = \frac{\partial T}{\partial x}(x, y) + \frac{\partial T}{\partial y}(x, y),$$

$$\frac{\partial T}{\partial z}(x, z) = \frac{\partial T}{\partial y}(x, y)$$

**Solution to Exercise 8**

By putting  $u$  for  $T$  in the solution to Exercise 7 we see that in terms of the variables  $x, z$  the equation takes the form

$$\frac{\partial u}{\partial x}(x, z) = 0.$$

**Solution to Exercise 9**

We see from Exercise 2 of subsection 0.2.2 that the general solution of

$$\frac{\partial u}{\partial x}(x, z) = 0$$

is

$$u(x, z) = f(z),$$

where  $f$  is an arbitrary function of one variable. Since  $y = x + z$ , we have  $z = y - x$ , so that the general solution of

$$\frac{\partial u}{\partial x}(x, y) + \frac{\partial u}{\partial y}(x, y) = 0$$

is

$$u(x, y) = f(y - x),$$

which agrees with the result suggested by Exercise 2 of subsection 0.2.3.

**Solution to Exercise 10**

We are asked to restate the general form of the chain rule given in the solution to Exercise 6 (with  $t$  being replaced by  $y$ ) in two different ways.

(i) As operator equations these are

$$\frac{\partial}{\partial x} = \frac{\partial \xi}{\partial x} \frac{\partial}{\partial \xi} + \frac{\partial \eta}{\partial x} \frac{\partial}{\partial \eta},$$

$$\frac{\partial}{\partial y} = \frac{\partial \xi}{\partial y} \frac{\partial}{\partial \xi} + \frac{\partial \eta}{\partial y} \frac{\partial}{\partial \eta}.$$

(ii) As a matrix equation, they can be expressed by

$$\begin{bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial u}{\partial y} \end{bmatrix} = \begin{bmatrix} \frac{\partial \xi}{\partial x} & \frac{\partial \eta}{\partial x} \\ \frac{\partial \xi}{\partial y} & \frac{\partial \eta}{\partial y} \end{bmatrix} \begin{bmatrix} \frac{\partial u}{\partial \xi} \\ \frac{\partial u}{\partial \eta} \end{bmatrix}.$$

**Solution to Exercise 11**

There are two methods of solution. The first involves performing the matrix multiplication explicitly and using the chain rule:

$$\begin{aligned}
 AB &= \begin{bmatrix} \frac{\partial \xi}{\partial x} & \frac{\partial \eta}{\partial x} & \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} \\ \frac{\partial \xi}{\partial y} & \frac{\partial \eta}{\partial y} & \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} \end{bmatrix} \\
 &= \begin{bmatrix} \left( \frac{\partial \xi}{\partial x} \frac{\partial x}{\partial \xi} + \frac{\partial \eta}{\partial x} \frac{\partial x}{\partial \eta} \right) & \left( \frac{\partial \xi}{\partial x} \frac{\partial y}{\partial \xi} + \frac{\partial \eta}{\partial x} \frac{\partial y}{\partial \eta} \right) \\ \left( \frac{\partial \xi}{\partial y} \frac{\partial x}{\partial \xi} + \frac{\partial \eta}{\partial y} \frac{\partial x}{\partial \eta} \right) & \left( \frac{\partial \xi}{\partial y} \frac{\partial y}{\partial \xi} + \frac{\partial \eta}{\partial y} \frac{\partial y}{\partial \eta} \right) \end{bmatrix},
 \end{aligned}$$

Using the solutions to Exercise 10(i) we see that this is equal to

$$\begin{bmatrix} \frac{\partial x}{\partial x} & \frac{\partial y}{\partial x} \\ \frac{\partial x}{\partial y} & \frac{\partial y}{\partial y} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

so that

$$AB = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

The second method is to note that corresponding to the matrix equation of Exercise 10(ii) giving  $\partial u/\partial x$ ,  $\partial u/\partial y$  in terms of  $\partial u/\partial \xi$  and  $\partial u/\partial \eta$ , there will be a similar equation giving  $\partial u/\partial \xi$ ,  $\partial u/\partial \eta$  in terms of  $\partial u/\partial x$  and  $\partial u/\partial y$ . The chain rule says that this equation is

$$\begin{bmatrix} \frac{\partial u}{\partial \xi} \\ \frac{\partial u}{\partial \eta} \end{bmatrix} = \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} \end{bmatrix} \begin{bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial u}{\partial y} \end{bmatrix}$$

and the  $2 \times 2$  matrix here is just  $B$ . By combining this equation with that of Exercise 10(ii) we see immediately that  $A$  and  $B$  must be inverse to each other, so that  $AB$  is the identity matrix. This result is the analogue in two dimensions of the one-dimensional formula

$$\frac{dy}{dx} \frac{dx}{dy} = 1;$$

but note that, for example,

$$\frac{\partial x}{\partial \xi} \frac{\partial \xi}{\partial x} \neq 1$$

in general, since the  $(1, 1)$  elements of two matrices inverse to each other are not normally reciprocals.

## 0S.3 SECOND-ORDER PARTIAL DIFFERENTIATION

### 0S.3.1 Second and Higher Order Partial Derivatives

#### Solution to Exercise 1

If  $u(x, t) = \sin xt + \sin x \sin t + t$  then

$$\frac{\partial u}{\partial x} = \boxed{t \cos xt + \cos x \sin t}$$

$$\frac{\partial u}{\partial t} = \boxed{x \cos xt + \sin x \cos t + 1}$$

$$\frac{\partial^2 u}{\partial x \partial t} = \boxed{\cos xt - xt \sin xt + \cos x \cos t}$$

$$\frac{\partial^2 u}{\partial t \partial x} = \boxed{\cos xt - xt \sin xt + \cos x \cos t}$$

$$\frac{\partial^2 u}{\partial t^2} = \boxed{-x^2 \sin xt - \sin x \sin t}$$

#### Solution to Exercise 2

Yes,

provided that the two mixed derivatives exist and are continuous. All the functions to be encountered in the course will satisfy these conditions.

#### Solution to Exercise 3

Since  $\partial^2/\partial x^2$  and  $\partial^2/\partial y^2$  commute, we have

$$\begin{aligned} \frac{\partial^{2n} f}{\partial x^{2n}} &= \left( \frac{\partial^2}{\partial x^2} \right)^n f \\ &= \left( \frac{\partial^2}{\partial x^2} \right)^{n-1} \left( \frac{\partial^2}{\partial x^2} \right) f \\ &= \left( \frac{\partial^2}{\partial x^2} \right)^{n-1} \left( \frac{\partial^2}{\partial y^2} \right) f \\ &= \frac{\partial^2}{\partial y^2} \left( \frac{\partial^2}{\partial x^2} \right)^{n-1} f \\ &= \frac{\partial^2}{\partial y^2} \left( \frac{\partial^2}{\partial x^2} \right)^{n-2} \left( \frac{\partial^2}{\partial y^2} \right) f \\ &= \left( \frac{\partial^2}{\partial y^2} \right)^2 \left( \frac{\partial^2}{\partial x^2} \right)^{n-2} f, \end{aligned}$$

and so on until all the  $\partial^2/\partial x^2$  have been changed into  $\partial^2/\partial y^2$ . Hence the result is

$$\left( \frac{\partial^2}{\partial x^2} \right)^n f = \left( \frac{\partial^2}{\partial y^2} \right)^n f,$$

or

$$\frac{\partial^{2n} f}{\partial x^{2n}} = \boxed{\frac{\partial^{2n} f}{\partial y^{2n}}}.$$

**Solution to Exercise 4**

This time we have

$$\left(\frac{\partial^2}{\partial x^2}\right)f = \left(-\frac{\partial^2}{\partial y^2}\right)f,$$

and so, the same procedure as we used in the preceding exercise gives

$$\left(\frac{\partial^2}{\partial x^2}\right)^n f = \left(-\frac{\partial^2}{\partial y^2}\right)^n f,$$

which can be written as

$$\frac{\partial^{2n} f}{\partial x^{2n}} = \boxed{(-1)^n \frac{\partial^{2n} f}{\partial y^{2n}}}.$$

**Solution to Exercise 5**

Proceeding as before we find that

$$\frac{\partial^n f}{\partial y^n} = \boxed{\frac{\partial^{2n} f}{\partial x^{2n}}}.$$

Hence

$$\frac{\partial^2 f}{\partial y^2} - 3 \frac{\partial^3 f}{\partial y \partial x^2} + 2 \frac{\partial^4 f}{\partial x^4} = \frac{\partial^2 f}{\partial y^2} - 3 \frac{\partial}{\partial y} \left( \frac{\partial f}{\partial y} \right) + 2 \frac{\partial^2 f}{\partial y^2},$$

giving

$$\frac{\partial^2 f}{\partial y^2} - 3 \frac{\partial^3 f}{\partial y \partial x^2} + 2 \frac{\partial^4 f}{\partial x^4} = \boxed{0}.$$

**0S.3.2 Solving Simple Second-Order Partial Differential Equations****Solution to Exercise 1**

Putting  $\partial u / \partial y = p$ , the equation becomes

$$\boxed{\frac{\partial p}{\partial x} = 0},$$

with solution

$$p(x, y) = \boxed{f(y)}$$

for an arbitrary function  $f$ . Since

$$\frac{\partial u}{\partial y}(x, y) = \boxed{f(y)}$$

has solution

$$u = \int f(y) + g(x),$$

the general solution of the original equation is

$$u(x, y) = \int f(y) + g(x)$$

where  $f$  and  $g$  are arbitrary functions of one variable. Since the equation we started from is symmetric in  $x$  and  $y$  we could equally well have solved the problem by putting  $\partial u/\partial x = p$  and following the same method thereafter. We would then have arrived at the general solution

$$u(x, y) = h(y) + \int g_1(x),$$

where  $h, g_1$  are arbitrary. It follows from this that we can express the general solution most simply as

$$u(x, y) = g(x) + h(y).$$

Since  $h$  is a primitive of  $f$  and  $g$  is a primitive of  $g_1$ , both  $h$  and  $g$  must be differentiable, but are otherwise arbitrary.

### Solution to Exercise 2

Putting  $\partial u/\partial x = p$ , the equation becomes

$$\frac{\partial p}{\partial x} = 0,$$

with solution

$$p(x, y) = f(y),$$

for arbitrary  $f$ . Since the equation

$$\frac{\partial u}{\partial x} = f(y)$$

has solution

$$u = xf(y) + g(y),$$

we have for the general solution of the original equation

$$u(x, y) = g(y) + xf(y),$$

where  $f$  and  $g$  are arbitrary functions of one variable.

### Solution to Exercise 3

Putting  $\partial u/\partial y = p$ , the equation becomes

$$\frac{\partial p}{\partial y} - \frac{1}{y}p = 0.$$

An integrating factor is

$$\begin{aligned} \exp \int \left( -\frac{1}{y} \right) dy &= \exp(-\ln y) \\ &= \frac{1}{y} \end{aligned}$$

(compare with Exercise 5 of subsection 0.2.2), and multiplying through by this factor brings the equation to the form

$$\frac{\partial}{\partial y} \left( \frac{p(x, y)}{y} \right) = 0,$$



which has solution

$$p(x, y) = \boxed{yf(x)},$$

where  $f$  is arbitrary. Since  $p = \partial u / \partial y$ , an integration with respect to  $y$  gives

$$u(x, y) = \boxed{\frac{1}{2}y^2f(x) + g(x)}.$$

The factor  $\frac{1}{2}$  may without loss of generality be absorbed into the arbitrary function  $f$ .

#### Solution to Exercise 4

In terms of  $p$ , where  $p = \partial u / \partial y$ , the equation is

$$x \frac{\partial p}{\partial x} = 2p,$$

or in normal form,

$$\frac{\partial p}{\partial x} - \frac{2}{x}p = 0.$$

An integrating factor is

$$\exp \int \left( -\frac{2}{x} \right) dx = \frac{1}{x^2},$$

bringing the equation to the form

$$\frac{\partial}{\partial x} \left( \frac{p(x, y)}{x^2} \right) = 0.$$

Integrating with respect to  $x$  and solving for  $p$  gives

$$p(x, y) = x^2 f(y)$$

with  $f$  arbitrary. Since  $p = \partial u / \partial y$  we can now integrate with respect to  $y$  to obtain

$$u(x, y) = \boxed{x^2 \int f(y) + g(x)}.$$

### 0S.3.3 Transforming Second-Order Derivatives

#### Solution to Exercise 1

From Exercise 6 of subsection 0.2.6,

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta},$$

$$\frac{\partial}{\partial t} = c \left( \frac{\partial}{\partial \xi} - \frac{\partial}{\partial \eta} \right).$$

Since  $\partial^2/\partial x^2 = (\partial/\partial x)^2$ ,  $\partial^2/\partial x \partial t = (\partial/\partial x)(\partial/\partial t)$  etc., we then have

$$\frac{\partial^2}{\partial x^2} = \frac{\partial^2}{\partial \xi^2} + 2 \frac{\partial^2}{\partial \xi \partial \eta} + \frac{\partial^2}{\partial \eta^2},$$

$$\frac{\partial^2}{\partial x \partial t} = c \left( \frac{\partial^2}{\partial \xi^2} - \frac{\partial^2}{\partial \eta^2} \right),$$

$$\frac{\partial^2}{\partial t^2} = c^2 \left( \frac{\partial^2}{\partial \xi^2} - 2 \frac{\partial^2}{\partial \xi \partial \eta} + \frac{\partial^2}{\partial \eta^2} \right).$$

#### Solution to Exercise 2

We see from Exercise 1 that

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial \xi^2} + 2 \frac{\partial^2 u}{\partial \xi \partial \eta} + \frac{\partial^2 u}{\partial \eta^2},$$

$$\frac{\partial^2 u}{\partial t^2} = c^2 \left( \frac{\partial^2 u}{\partial \xi^2} - 2 \frac{\partial^2 u}{\partial \xi \partial \eta} + \frac{\partial^2 u}{\partial \eta^2} \right).$$

Subtracting the second equation from  $c^2$  times the first, we have

$$c^2 \frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial t^2} = 4c^2 \frac{\partial^2 u}{\partial \xi \partial \eta},$$

so that

$$\frac{\partial^2 u}{\partial \xi \partial \eta} = 0.$$

By reference to the solution of Exercise 1 in Subsection 0.3.2 we see that the general solution of this equation is

$$u(\xi, \eta) = f(\xi) + g(\eta),$$

for arbitrary differentiable functions  $f$  and  $g$ , so that the solution of the wave equation is

$$u(x, t) = f(x + ct) + g(x - ct).$$

**Solution to Exercise 3**

Using the chain rule,

$$\frac{\partial}{\partial x} = \frac{\partial \xi}{\partial x} \frac{\partial}{\partial \xi} + \frac{\partial \eta}{\partial x} \frac{\partial}{\partial \eta},$$

$$\frac{\partial}{\partial t} = \frac{\partial \xi}{\partial t} \frac{\partial}{\partial \xi} + \frac{\partial \eta}{\partial t} \frac{\partial}{\partial \eta}.$$

Since the given expressions for  $\xi, \eta$  mean that

$$\frac{\partial \xi}{\partial x} = \alpha, \frac{\partial \xi}{\partial t} = \beta, \frac{\partial \eta}{\partial x} = \gamma, \frac{\partial \eta}{\partial t} = \delta,$$

we have

$$\frac{\partial}{\partial x} = \alpha \frac{\partial}{\partial \xi} + \gamma \frac{\partial}{\partial \eta},$$

$$\frac{\partial}{\partial t} = \beta \frac{\partial}{\partial \xi} + \delta \frac{\partial}{\partial \eta}.$$

The second-order derivatives are, therefore,

$$\frac{\partial^2}{\partial x^2} = \alpha^2 \frac{\partial^2}{\partial \xi^2} + 2\alpha\gamma \frac{\partial^2}{\partial \xi \partial \eta} + \gamma^2 \frac{\partial^2}{\partial \eta^2},$$

$$\frac{\partial^2}{\partial x \partial t} = \alpha\beta \frac{\partial^2}{\partial \xi^2} + (\alpha\delta + \beta\gamma) \frac{\partial^2}{\partial \xi \partial \eta} + \gamma\delta \frac{\partial^2}{\partial \eta^2},$$

$$\frac{\partial^2}{\partial t^2} = \beta^2 \frac{\partial^2}{\partial \xi^2} + 2\beta\delta \frac{\partial^2}{\partial \xi \partial \eta} + \delta^2 \frac{\partial^2}{\partial \eta^2},$$

so that

$$L = \left( A\beta^2 + B\alpha\beta + C\alpha^2 \right) \frac{\partial^2}{\partial \xi^2} + (2A\beta\delta + B(\alpha\delta + \beta\gamma) + 2C\alpha\gamma) \frac{\partial^2}{\partial \xi \partial \eta} + (A\delta^2 + B\gamma\delta + C\gamma^2) \frac{\partial^2}{\partial \eta^2}$$

**Solution to Exercise 4**

By either of the ways suggested we get

$$\frac{\partial}{\partial r} = \cos \theta \frac{\partial}{\partial x} + \sin \theta \frac{\partial}{\partial y},$$

$$\frac{\partial}{\partial \theta} = -r \sin \theta \frac{\partial}{\partial x} + r \cos \theta \frac{\partial}{\partial y}.$$

**Solution to Exercise 5**

We have

$$\begin{bmatrix} \frac{\partial}{\partial r} \\ \frac{\partial}{\partial \theta} \end{bmatrix} = \begin{bmatrix} \cos \theta & \sin \theta \\ -r \sin \theta & r \cos \theta \end{bmatrix} \begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{bmatrix}.$$

The matrix of coefficients can be inverted in the usual way:

$$\begin{bmatrix} \cos \theta & \sin \theta & 1 & 0 \\ -r \sin \theta & r \cos \theta & 0 & 1 \end{bmatrix}$$

Stage 1: Multiply first row by  $\sec \theta$  and then add  $r \sin \theta$  times the first row to the second row. This gives

$$\begin{bmatrix} 1 & \tan \theta & \sec \theta & 0 \\ 0 & r \sec \theta & r \tan \theta & 1 \end{bmatrix}.$$

Stage 2: Multiply second row by  $(1/r) \cos \theta$  and then subtract  $\tan \theta$  times the second row from the first row. This gives

$$\begin{bmatrix} 1 & 0 & \cos \theta & -\frac{1}{r} \sin \theta \\ 0 & 1 & \sin \theta & \frac{1}{r} \cos \theta \end{bmatrix}.$$

Reading off the inverse of the matrix we started with,

$$\begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{bmatrix} = \begin{bmatrix} \cos \theta & -\frac{1}{r} \sin \theta \\ \sin \theta & \frac{1}{r} \cos \theta \end{bmatrix} \begin{bmatrix} \frac{\partial}{\partial r} \\ \frac{\partial}{\partial \theta} \end{bmatrix},$$

or in other words,

$$\begin{aligned} \frac{\partial}{\partial x} &= \cos \theta \frac{\partial}{\partial r} - \frac{1}{r} \sin \theta \frac{\partial}{\partial \theta}, \\ \frac{\partial}{\partial y} &= \sin \theta \frac{\partial}{\partial r} + \frac{1}{r} \cos \theta \frac{\partial}{\partial \theta}. \end{aligned}$$

#### Solution to Exercise 6

$$\begin{aligned} \frac{\partial^2}{\partial x^2} &= \left( \cos \theta \frac{\partial}{\partial r} - \frac{1}{r} \sin \theta \frac{\partial}{\partial \theta} \right) \left( \cos \theta \frac{\partial}{\partial r} - \frac{1}{r} \sin \theta \frac{\partial}{\partial \theta} \right) \\ &= \cos^2 \theta \frac{\partial^2}{\partial r^2} - \cos \theta \frac{\partial}{\partial r} \left( \frac{1}{r} \sin \theta \frac{\partial}{\partial \theta} \right) - \frac{1}{r} \sin \theta \frac{\partial}{\partial \theta} \left( \cos \theta \frac{\partial}{\partial r} \right) \\ &\quad + \frac{1}{r} \sin \theta \frac{\partial}{\partial \theta} \left( \frac{1}{r} \sin \theta \frac{\partial}{\partial \theta} \right) \\ &= \cos^2 \theta \frac{\partial^2}{\partial r^2} + \frac{\sin \theta \cos \theta}{r^2} \frac{\partial}{\partial \theta} - \frac{\sin \theta \cos \theta}{r} \frac{\partial^2}{\partial r \partial \theta} + \frac{\sin^2 \theta}{r} \frac{\partial}{\partial r} \\ &\quad - \frac{\sin \theta \cos \theta}{r} \frac{\partial^2}{\partial r \partial \theta} + \frac{\sin \theta \cos \theta}{r^2} \frac{\partial}{\partial \theta} + \frac{\sin^2 \theta}{r^2} \frac{\partial^2}{\partial \theta^2}. \end{aligned}$$

Therefore

$$\frac{\partial^2}{\partial x^2} = \cos^2 \theta \frac{\partial^2}{\partial r^2} - \frac{2 \sin \theta \cos \theta}{r} \frac{\partial^2}{\partial r \partial \theta} + \frac{\sin^2 \theta}{r^2} \frac{\partial^2}{\partial \theta^2} + \frac{\sin^2 \theta}{r} \frac{\partial}{\partial r} + \frac{2 \sin \theta \cos \theta}{r^2} \frac{\partial}{\partial \theta}.$$

Similarly,

$$\begin{aligned} \frac{\partial^2}{\partial y^2} &= \left( \sin \theta \frac{\partial}{\partial r} + \frac{1}{r} \cos \theta \frac{\partial}{\partial \theta} \right) \left( \sin \theta \frac{\partial}{\partial r} + \frac{1}{r} \cos \theta \frac{\partial}{\partial \theta} \right) \\ &= \sin^2 \theta \frac{\partial^2}{\partial r^2} + \sin \theta \frac{\partial}{\partial r} \left( \frac{1}{r} \cos \theta \frac{\partial}{\partial \theta} \right) + \frac{1}{r} \cos \theta \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial}{\partial r} \right) \\ &\quad + \frac{1}{r} \cos \theta \frac{\partial}{\partial \theta} \left( \frac{1}{r} \cos \theta \frac{\partial}{\partial \theta} \right) \end{aligned}$$

$$\begin{aligned}
&= \sin^2 \theta \frac{\partial^2}{\partial r^2} - \frac{\sin \theta \cos \theta}{r^2} \frac{\partial}{\partial \theta} + \frac{\sin \theta \cos \theta}{r} \frac{\partial^2}{\partial r \partial \theta} + \frac{\cos^2 \theta}{r} \frac{\partial}{\partial r} \\
&\quad + \frac{\sin \theta \cos \theta}{r} \frac{\partial^2}{\partial r \partial \theta} - \frac{\sin \theta \cos \theta}{r^2} \frac{\partial}{\partial \theta} + \frac{\cos^2 \theta}{r^2} \frac{\partial^2}{\partial \theta^2}.
\end{aligned}$$

Therefore,

$$\frac{\partial^2}{\partial y^2} = \boxed{\sin^2 \theta \frac{\partial^2}{\partial r^2} + \frac{2 \sin \theta \cos \theta}{r} \frac{\partial^2}{\partial r \partial \theta} + \frac{\cos^2 \theta}{r^2} \frac{\partial^2}{\partial \theta^2} + \frac{\cos^2 \theta}{r} \frac{\partial}{\partial r} - \frac{2 \sin \theta \cos \theta}{r^2} \frac{\partial}{\partial \theta}}.$$

It follows that

$$\nabla^2 = \boxed{\frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2}}.$$

### Solution to Exercise 7

(i) Since

$$\frac{\partial \xi}{\partial x} = 2x, \frac{\partial \eta}{\partial x} = 0, \frac{\partial \xi}{\partial t} = 2t, \frac{\partial \eta}{\partial t} = 1,$$

we have by the chain rule,

$$\frac{\partial}{\partial x} = \boxed{2x \frac{\partial}{\partial \xi}},$$

$$\frac{\partial}{\partial t} = \boxed{2t \frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta}}.$$

(ii) From the first answer in part (i), we have

$$\frac{\partial}{\partial \eta} = \boxed{\frac{1}{2x} \frac{\partial}{\partial x}}.$$

Substituting this into the second answer in part (i) gives

$$\frac{\partial}{\partial \eta} = \boxed{\frac{\partial}{\partial t} - \frac{t}{x} \frac{\partial}{\partial x}}.$$

(iii)

$$\begin{aligned}
\frac{\partial^2}{\partial \eta^2} &= \left( \frac{\partial}{\partial t} - \frac{t}{x} \frac{\partial}{\partial x} \right) \left( \frac{\partial}{\partial t} - \frac{t}{x} \frac{\partial}{\partial x} \right) \\
&= \frac{\partial^2}{\partial t^2} - \frac{\partial}{\partial t} \left( \frac{t}{x} \frac{\partial}{\partial x} \right) - \frac{t}{x} \frac{\partial^2}{\partial x \partial t} + \frac{t}{x} \frac{\partial}{\partial x} \left( \frac{t}{x} \frac{\partial}{\partial x} \right) \\
&= \frac{\partial^2}{\partial t^2} - \frac{1}{x} \frac{\partial}{\partial x} - \frac{t}{x} \frac{\partial^2}{\partial x \partial t} - \frac{t}{x} \frac{\partial^2}{\partial x \partial t} - \frac{t^2}{x^3} \frac{\partial}{\partial x} + \frac{t^2}{x^2} \frac{\partial^2}{\partial x^2},
\end{aligned}$$

so that we have

$$\frac{\partial^2}{\partial \eta^2} = \frac{1}{x^2} L - \boxed{\frac{1}{x} \left( 1 + \frac{t^2}{x^2} \right) \frac{\partial}{\partial x}}.$$

(iv) The last equation in part (iii) can be written as

$$L = x^2 \frac{\partial^2}{\partial \eta^2} + x \left( 1 + \frac{t^2}{x^2} \right) \frac{\partial}{\partial x},$$

and using the first answer in part (i) gives

$$L = \left[ x^2 \frac{\partial^2}{\partial \eta^2} + 2(x^2 + t^2) \frac{\partial}{\partial \xi} \right].$$

### Solution to Exercise 8

Using the solution to Exercise 7(i), we see that the right-hand side of the given equation can be expressed as

$$\frac{x^2}{t} \left( 2t \frac{\partial u}{\partial \xi} + \frac{\partial u}{\partial \eta} \right) + \frac{t^2}{x} \left( 2x \frac{\partial u}{\partial \xi} \right) = 2(x^2 + t^2) \frac{\partial u}{\partial \xi} + \frac{x^2}{t} \frac{\partial u}{\partial \eta},$$

so that upon using the expression for  $L$  derived in Exercise 7(iv), dividing through by  $x^2$  and replacing  $t$  by  $\eta$ , the transformed equation is seen to be

$$\left[ \frac{\partial^2 u}{\partial \eta^2} = \frac{1}{\eta} \frac{\partial u}{\partial \eta} \right].$$

By reference to Exercise 3 of subsection 0.3.2 we see that the general solution of the transformed equation is

$$u(\xi, \eta) = \left[ f(\xi) + \eta^2 g(\xi) \right],$$

where  $f$  and  $g$  are arbitrary functions of one variable. Thus the general solution of the original equation is

$$u(x, t) = \left[ f(x^2 + t^2) + t^2 g(x^2 + t^2) \right].$$

### Solution to Exercise 9

In Exercise 7 of subsection 0.1.1 we saw that

$$\frac{d}{dx}(\arctan x) = \frac{1}{1 + x^2},$$

and using this result in conjunction with the composite function rule we get

$$\frac{d}{dx} \left( \frac{1}{2} \arctan \frac{1}{2} x \right) = \frac{1}{4 + x^2},$$

so that we have

$$\frac{\partial \xi}{\partial x} = \frac{1}{1 + x^2}, \quad \frac{\partial \xi}{\partial t} = -1,$$

$$\frac{\partial \eta}{\partial x} = \frac{1}{4 + x^2}, \quad \frac{\partial \eta}{\partial t} = -1.$$

The chain rule then gives

$$\frac{\partial}{\partial x} = \frac{1}{1 + x^2} \frac{\partial}{\partial \xi} + \frac{1}{4 + x^2} \frac{\partial}{\partial \eta},$$

$$\frac{\partial}{\partial t} = - \left( \frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta} \right).$$

Solving these equations for  $\partial/\partial \xi$  and  $\partial/\partial \eta$  we find

$$\frac{\partial}{\partial \xi} = \frac{1}{3}(1 + x^2)(4 + x^2) \left( \frac{\partial}{\partial x} + \frac{1}{4 + x^2} \frac{\partial}{\partial t} \right),$$

$$\frac{\partial}{\partial \eta} = -\frac{1}{3}(1 + x^2)(4 + x^2) \left( \frac{\partial}{\partial x} + \frac{1}{1 + x^2} \frac{\partial}{\partial t} \right).$$

It follows that

$$\begin{aligned}
 \frac{\partial^2}{\partial \xi \partial \eta} &= -\frac{1}{9}(1+x^2)(4+x^2) \left( \frac{\partial}{\partial x} + \frac{1}{4+x^2} \frac{\partial}{\partial t} \right) \\
 &\quad \times \left\{ (1+x^2)(4+x^2) \left( \frac{\partial}{\partial x} + \frac{1}{1+x^2} \frac{\partial}{\partial t} \right) \right\} \\
 &= -\frac{1}{9}(1+x^2)(4+x^2) \left\{ \frac{\partial^2}{\partial t^2} + (5+2x^2) \frac{\partial^2}{\partial x \partial t} + (1+x^2)(4+x^2) \frac{\partial^2}{\partial x^2} \right. \\
 &\quad \left. + 2x(5+2x^2) \frac{\partial}{\partial x} + 2x \frac{\partial}{\partial t} \right\} \\
 &= -\frac{1}{9}(1+x^2)(4+x^2) \left\{ L + 2x \left[ (5+2x^2) \frac{\partial}{\partial x} + \frac{\partial}{\partial t} \right] \right\}.
 \end{aligned}$$

Using the expressions above for  $\partial/\partial x$ ,  $\partial/\partial t$  in terms of  $\partial/\partial \xi$ ,  $\partial/\partial \eta$ ,

$$(5+2x^2) \frac{\partial}{\partial x} + \frac{\partial}{\partial t} = \frac{4+x^2}{1+x^2} \frac{\partial}{\partial \xi} + \frac{1+x^2}{4+x^2} \frac{\partial}{\partial \eta},$$

so that the expression for  $L$  is

$$L = \frac{-9}{(1+x^2)(4+x^2)} \left[ \frac{\partial^2}{\partial \xi \partial \eta} + \frac{2x}{9}(4+x^2)^2 \frac{\partial}{\partial \xi} + \frac{2x}{9}(1+x^2)^2 \frac{\partial}{\partial \eta} \right],$$

which is equivalent to the formula for  $L[u]$  given at the top of page 46 in  $W$ .

## 0S.4 VECTOR ALGEBRA AND LINE INTEGRALS

### 0S.4.1 Vector Algebra

#### Solution to Exercise 1

- (i) If we complete the parallelogram which has (arrows belonging to)  $\mathbf{a}$  and  $\mathbf{b}$  as adjacent sides, then the diagonal from the point  $O$  where the sides  $\mathbf{a}$  and  $\mathbf{b}$  meet is  $\mathbf{a} + \mathbf{b}$ . Since the diagonals of a parallelogram bisect each other,  $\frac{1}{2}(\mathbf{a} + \mathbf{b})$  will be the position vector of a point halfway between the two vertices of the parallelogram with position vectors  $\mathbf{a}$  and  $\mathbf{b}$  (taking  $O$  as the origin). The answer is therefore

$$\frac{1}{2}(\mathbf{a} + \mathbf{b})$$

- (ii) The answer here depends on how the arrows are arranged. If  $\mathbf{b}$  points from the end of the arrow  $\mathbf{a}$  (or vice versa) then the third side of the triangle is

$$\mathbf{a} + \mathbf{b}$$

or  $-\mathbf{a} - \mathbf{b}$ , depending upon how the vector is directed along the side. If  $\mathbf{a}$  and  $\mathbf{b}$  have corresponding ends coincident then the third side of the triangle will be  $\mathbf{a} - \mathbf{b}$  or  $\mathbf{b} - \mathbf{a}$ , depending on what direction is associated with it.

- (iii) Since by the definition of the scalar product,

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}| |\mathbf{b}| \cos \theta,$$

where  $\theta$  is the angle between  $\mathbf{a}$  and  $\mathbf{b}$ , the required cosine is equal to

$$\frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}| |\mathbf{b}|}$$

- (iv) The projection of  $\mathbf{a}$  along the direction of  $\mathbf{b}$  is

$$|\mathbf{a}| \cos \theta,$$

which, we see from part (iii), is equal to

$$\frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{b}|}$$

- (v) Since  $\mathbf{a} \times \mathbf{b}$  is perpendicular to both  $\mathbf{a}$  and  $\mathbf{b}$  the required unit vector is

$$\frac{\mathbf{a} \times \mathbf{b}}{|\mathbf{a} \times \mathbf{b}|}$$

- (vi) The parallelogram's area is  $|\mathbf{a}| |\mathbf{b}| \sin \theta$ , but we have  $|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}| |\mathbf{b}| \sin \theta$  by definition. The area is therefore

$$|\mathbf{a} \times \mathbf{b}|$$

- (vii)  $\mathbf{a}$  and  $\mathbf{b}$  are perpendicular if  $\theta = \pi/2$ , or  $\cos \theta = 0$ , or

$$\mathbf{a} \cdot \mathbf{b} = 0$$

- (viii)  $\mathbf{a}$  and  $\mathbf{b}$  are parallel if  $\theta = 0$ , or  $\sin \theta = 0$ , or

$$\mathbf{a} \times \mathbf{b} = \mathbf{0}$$



**Solution to Exercise 2**

- (i) If
- $\mathbf{a} = \mathbf{i} + \mathbf{j}$
- , then
- $|\mathbf{a}| = \sqrt{1^2 + 1^2}$
- , or,

$$|\mathbf{a}| = \boxed{\sqrt{2}}.$$

- (ii)
- $\mathbf{a} - 2\mathbf{b} = -3\mathbf{i} + 3\mathbf{j} + 2\mathbf{k}$
- ,

so that

$$|\mathbf{a} - 2\mathbf{b}| = \sqrt{3^2 + 3^2 + 2^2},$$

or

$$|\mathbf{a} - 2\mathbf{b}| = \boxed{\sqrt{22}}.$$

- (iii)
- $\mathbf{a} \cdot \mathbf{b} = (\mathbf{i} + \mathbf{j}) \cdot (2\mathbf{i} - \mathbf{j} - \mathbf{k}) = 2 - 1$
- ,

so that

$$\mathbf{a} \cdot \mathbf{b} = \boxed{1}.$$

- (iv)
- $\mathbf{a} \times \mathbf{b} = (\mathbf{i} + \mathbf{j}) \times (2\mathbf{i} - \mathbf{j} - \mathbf{k})$
- .

Using the formulae for vector products involving the unit vectors,  $\mathbf{i}$ ,  $\mathbf{j}$  and  $\mathbf{k}$  (see page 8 of *SS* if necessary) we see that this is equal to

$$-\mathbf{i} \times \mathbf{j} - \mathbf{i} \times \mathbf{k} + \mathbf{j} \times 2\mathbf{i} - \mathbf{j} \times \mathbf{k} = -\mathbf{k} + \mathbf{j} - 2\mathbf{k} - \mathbf{i},$$

so that

$$\mathbf{a} \times \mathbf{b} = \boxed{-\mathbf{i} + \mathbf{j} - 3\mathbf{k}}.$$

- (v)
- $\mathbf{a} \cdot \mathbf{b} \times \mathbf{c} = (\mathbf{i} + \mathbf{j}) \cdot (2\mathbf{i} - \mathbf{j} - \mathbf{k}) \times (-\mathbf{i} + 3\mathbf{j} + \mathbf{k})$
- .

Proceeding as in part (iv),

$$\begin{aligned} (2\mathbf{i} - \mathbf{j} - \mathbf{k}) \times (-\mathbf{i} + 3\mathbf{j} + \mathbf{k}) \\ &= 2\mathbf{i} \times 3\mathbf{j} + 2\mathbf{i} \times \mathbf{k} - \mathbf{j} \times (-\mathbf{i}) - \mathbf{j} \times \mathbf{k} - \mathbf{k} \times (-\mathbf{i}) - \mathbf{k} \times 3\mathbf{j} \\ &= 6\mathbf{k} - 2\mathbf{j} - \mathbf{k} - \mathbf{i} + \mathbf{j} + 3\mathbf{i} \\ &= 2\mathbf{i} - \mathbf{j} + 5\mathbf{k}. \end{aligned}$$

Then

$$(\mathbf{i} + \mathbf{j}) \cdot (2\mathbf{i} - \mathbf{j} + 5\mathbf{k}) = 2 - 1,$$

so that

$$\mathbf{a} \cdot \mathbf{b} \times \mathbf{c} = \boxed{1}.$$

**Solution to Exercise 3**

- (i) The velocity
- $\mathbf{v}$
- of the particle is equal to
- $d\mathbf{r}/dt$
- . This is

$$\boxed{-6 \sin 3t \mathbf{i} + 6 \cos 3t \mathbf{j} + 2\mathbf{k}}.$$

- (ii) The speed is the magnitude of the velocity. Since

$$|\mathbf{v}| = \sqrt{36 \sin^2 3t + 36 \cos^2 3t + 4},$$

the speed is

$$\boxed{\sqrt{40}}.$$

- (iii) Since the particle is travelling in the direction of its velocity vector at any instant, the unit vector required is  $\mathbf{v}/|\mathbf{v}|$ . From parts (i) and (ii) this is

$$\frac{1}{\sqrt{40}}(-6 \sin 3t \mathbf{i} + 6 \cos 3t \mathbf{j} + 2 \mathbf{k})$$

## 0S.4.2 Line Integrals

### Solution to Exercise 1

Since

$$\begin{aligned}\mathbf{F} \cdot d\mathbf{r} &= (x\mathbf{i} + x^2y\mathbf{j}) \cdot (\mathbf{i} dx + \mathbf{j} dy) \\ &= x dx + x^2y dy,\end{aligned}$$

and since  $dy = 0$  on the line segment from  $(-1, 0)$  to  $(1, 0)$ ,

$$\begin{aligned}\int_C \mathbf{F} \cdot d\mathbf{r} &= \int_{-1}^1 x dx \\ &= \left[\frac{1}{2}x^2\right]_{-1}^1,\end{aligned}$$

so that

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \boxed{0}.$$

### Solution to Exercise 2

Clearly, from the answer to Exercise 1, the integral is equal to the integral along the line segment from  $(1, 0)$  to  $(1, 1)$ . Since  $x = 1$  and  $dx = 0$  here,

$$\begin{aligned}\int_C \mathbf{F} \cdot d\mathbf{r} &= \int_0^1 y dy \\ &= \left[\frac{1}{2}y^2\right]_0^1.\end{aligned}$$

so that

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \boxed{\frac{1}{2}}.$$

### Solution to Exercise 3

Here we have

$$\frac{dx}{dt} = 2t, \frac{dy}{dt} = 3t^2$$

along  $C$ , so that

$$\begin{aligned}\int_C \mathbf{F} \cdot d\mathbf{r} &= \int_C x dx + \int_C x^2y dy \\ &= \int_0^1 t^2 \cdot 2t dt + \int_0^1 t^4 \cdot t^3 \cdot 3t^2 dt \\ &= \int_0^1 (2t^3 + 3t^9) dt \\ &= \left[\frac{1}{2}t^4 + \frac{3}{10}t^{10}\right]_0^1,\end{aligned}$$

so that

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \boxed{\frac{4}{5}}.$$

**Solution to Exercise 4**

The semicircle is given in Cartesian coordinates by

$$x^2 + y^2 = 1, y \geq 0.$$

We see that  $x^2 + y^2$  will be equal to unity if we put

$$x = \cos t, y = \sin t,$$

and that  $t$  will vary from 0 to  $\pi$  as the point  $(x, y)$  moves around the semicircle from  $(1, 0)$  to  $(-1, 0)$ . Thus our parametric representation of the semicircle is

$$x = \boxed{\cos t},$$

$$y = \boxed{\sin t},$$

$$t \in \boxed{[0, \pi]}.$$

**Solution to Exercise 5**

Here we have

$$\begin{aligned} \mathbf{F} \cdot d\mathbf{r} &= (\mathbf{i} + y\mathbf{j}) \cdot (\mathbf{i} dx + \mathbf{j} dy) \\ &= dx + y dy, \end{aligned}$$

so that

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C dx + \int_C y dy.$$

On the semicircle,

$$\frac{dx}{dt} = -\sin t, \frac{dy}{dt} = \cos t,$$

so that

$$\begin{aligned} \int_C \mathbf{F} \cdot d\mathbf{r} &= -\int_0^\pi \sin t dt + \int_0^\pi \sin t \cos t dt \\ &= [\cos t - \tfrac{1}{2} \cos 2t]_0^\pi, \end{aligned}$$

or

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \boxed{-2}.$$

**Solution to Exercise 6**

We have

$$\mathbf{v} \times d\mathbf{r} = (xy dy - y^2 dx)\mathbf{k},$$

so that using the parametrization of Exercise 4 for the semicircle,

$$\begin{aligned} \int_C \mathbf{v} \times d\mathbf{r} &= \left\{ \int_C xy dy - \int_C y^2 dx \right\} \mathbf{k} \\ &= \left\{ \int_0^\pi \cos t \sin t \cos t dt - \int_0^\pi \sin^2 t (-\sin t) dt \right\} \mathbf{k} \\ &= \left\{ \int_0^\pi \sin t dt \right\} \mathbf{k} \\ &= [-\cos t]_0^\pi \mathbf{k}, \end{aligned}$$

so that

$$\int_C \mathbf{v} \times d\mathbf{r} = \boxed{2\mathbf{k}}.$$

Since

$$\int_C \mathbf{v} \times d\mathbf{r} = \mathbf{k} \int_C \mathbf{v} \cdot \mathbf{n} \, dr,$$

it follows that

$$\int_C \mathbf{v} \cdot \mathbf{n} \, dr = \boxed{2}.$$

0S.5 DOUBLE AND VOLUME INTEGRALS

0S.5.1 Double Integrals

Solution to Exercise 1

The region of integration is enclosed between the lines  $x = 0$ ,  $x = 1$ ,  $y = 0$  and  $y = 2x + 1$ . A diagram of this region can be found on page 17 of *Unit MST 282 6*, or on page 11 of *Unit 3*.

Solution to Exercise 2

$$\begin{aligned}\int_0^{2x+1} xy \, dy &= x \int_0^{2x+1} y \, dy \\ &= x \left[ \frac{1}{2} y^2 \right]_0^{2x+1},\end{aligned}$$

that is,

$$\int_0^{2x+1} xy \, dy = \boxed{\frac{1}{2}x(2x + 1)^2}.$$

Solution to Exercise 3

$$\begin{aligned}\int_0^1 \frac{1}{2}x(2x + 1)^2 \, dx &= \frac{1}{2} \int_0^1 (4x^3 + 4x^2 + x) \, dx \\ &= \frac{1}{2} \left[ x^4 + \frac{4}{3}x^3 + \frac{1}{2}x^2 \right]_0^1 \\ &= \frac{1}{2} \left[ 1 + \frac{4}{3} + \frac{1}{2} \right],\end{aligned}$$

so that

$$\int_0^1 \left[ \int_0^{2x+1} xy \, dy \right] dx = \boxed{\frac{17}{12}}.$$

Solution to Exercise 4

The “inner” integration is performed with respect to  $y$ . Since the ellipse can be written as

$$y = \pm b\sqrt{1 - \frac{x^2}{a^2}},$$

the region enclosed by the ellipse can be described by

$$\begin{aligned}-b\sqrt{1 - \frac{x^2}{a^2}} &\leq y \leq b\sqrt{1 - \frac{x^2}{a^2}}, \\ -a &\leq x \leq a.\end{aligned}$$

The limits (end-points) of the two integrations are then the upper and lower bounds in these two inequalities. Therefore:

$$\begin{aligned}\text{Limits of } y\text{-integration are } &\int_{-b\sqrt{1 - x^2/a^2}}^{b\sqrt{1 - x^2/a^2}}; \\ \text{Limits of } x\text{-integration are } &\int_{-a}^a;\end{aligned}$$

Resulting double integral is

$$\int_{-a}^a \left[ \int_{-b\sqrt{1-x^2/a^2}}^{b\sqrt{1-x^2/a^2}} \left( 1 - \frac{x^2}{a^2} - \frac{y^2}{b^2} \right) dy \right] dx$$

### Solution to Exercise 5

$$\begin{aligned} \int_{-b\sqrt{1-x^2/a^2}}^{b\sqrt{1-x^2/a^2}} \left( 1 - \frac{x^2}{a^2} - \frac{y^2}{b^2} \right) dy \\ &= \left[ \left( 1 - \frac{x^2}{a^2} \right) y - \frac{y^3}{3b^2} \right]_{-b\sqrt{1-x^2/a^2}}^{b\sqrt{1-x^2/a^2}} \\ &= \left[ 1 - \frac{x^2}{a^2} - \frac{1}{3} \left( 1 - \frac{x^2}{a^2} \right) \right] 2b\sqrt{1 - \frac{x^2}{a^2}}, \end{aligned}$$

so that

Inner integral is  $\frac{4}{3}b \left( 1 - \frac{x^2}{a^2} \right)^{\frac{3}{2}}$ .

We are left with the integral

$$\int_{-a}^a \frac{4}{3}b \left( 1 - \frac{x^2}{a^2} \right)^{\frac{3}{2}} dx$$

to evaluate. The substitution to make here is  $x = a \sin v$ . (You may like to refer to section III.3 of *TI* to see why we choose this particular substitution. We can see that our integral is of the form given at the beginning of that section, with  $f(x) = 0$ ,  $h(x) = \frac{4}{3}b(1 - x^2/a^2)$ ,  $ax^2 + bx + c$  replaced by  $1 - x^2/a^2$ ,  $g(x) = 1$ ,  $k(x) = 0$ . Subsection III.3.2(iii) gives the substitution we are using.) The range  $-a \leq x \leq a$  is equivalent to  $-\pi/2 \leq v \leq \pi/2$ , and  $dx/dv = a \cos v$ . Thus the integral becomes

$$\frac{4}{3}ba \int_{-\pi/2}^{\pi/2} \cos^4 v \, dv = \frac{1}{2}\pi ab,$$

using the information given in the question. Thus:

Final result is  $\frac{1}{2}\pi ab$ .

### Solution to Exercise 6

Putting  $x^2 + y^2 = r^2$  and replacing  $dx \, dy$  by  $r \, dr \, d\theta$ ,

$$\iint_D \left( 1 - \frac{x^2 + y^2}{a^2} \right) dx \, dy = \iint_D \left( 1 - \frac{r^2}{a^2} \right) r \, dr \, d\theta.$$

$D$  is composed of points  $(r, \theta)$ , where

$$0 \leq r \leq a,$$

$$-\pi \leq \theta \leq \pi.$$

Therefore, the double integral in polar coordinates is

$$\int_0^a \left[ \int_{-\pi}^{\pi} \left( 1 - \frac{r^2}{a^2} \right) r \, d\theta \right] dr$$

The inner integral is equal to

$$\left( 1 - \frac{r^2}{a^2} \right) r \int_{-\pi}^{\pi} d\theta = 2\pi r \left( 1 - \frac{r^2}{a^2} \right).$$

The outer integral is

$$\begin{aligned} 2\pi \int_0^a \left( r - \frac{r^3}{a^2} \right) dr &= 2\pi \left[ \frac{1}{2}r^2 - \frac{r^4}{4a^2} \right]_0^a \\ &= 2\pi \left[ \frac{1}{2}a^2 - \frac{1}{4}a^2 \right], \end{aligned}$$

so that the value of the integral is

$$\frac{1}{2}\pi a^2.$$

This agrees with the result of Exercise 5 when  $b = a$ .

## 0S.5.2 Volume Integrals

### Solution to Exercise 1

The interior of the cube consists of points  $(x, y, z)$  for which

$$0 \leq x \leq 1,$$

$$0 \leq y \leq 1,$$

$$0 \leq z \leq 1.$$

Thus the triple integral is

$$\int_0^1 \left\{ \int_0^1 \left[ \int_0^1 (x + y + z) dx \right] dy \right\} dz.$$

Since

$$\int_0^1 (x + y + z) dx = \left[ \frac{1}{2}x^2 + (y + z)x \right]_0^1,$$

the inner integral is

$$\frac{1}{2} + y + z.$$

Since

$$\int_0^1 \left( \frac{1}{2} + y + z \right) dy = \left[ \left( \frac{1}{2} + z \right)y + \frac{1}{2}y^2 \right]_0^1,$$

the next integral is

$$1 + z.$$

Since

$$\int_0^1 (1 + z) dz = \left[ z + \frac{1}{2}z^2 \right]_0^1,$$

the outer integral is

$$\frac{3}{2}.$$

**Solution to Exercise 2**

The interior of the prism consists of points  $(x, y, z)$  for which

$$0 \leq x \leq 1$$

$$0 \leq y \leq 1 - z$$

$$0 \leq z \leq 1,$$

so that the triple integral is

$$\left. \int_{\boxed{0}}^{\boxed{1}} \left\{ \int_{\boxed{0}}^{\boxed{1-z}} \left[ \int_{\boxed{0}}^{\boxed{1}} 3x^2 yz \, dx \right] dy \right\} dz.$$

**Solution to Exercise 3**

$$\int_0^1 3x^2 yz \, dx = [x^3 yz]_0^1,$$

so that the inner integral is

$$\boxed{yz}.$$

$$\int_0^{1-z} yz \, dy = [\tfrac{1}{2}y^2 z]_0^{1-z},$$

so that the next integral is

$$\boxed{\tfrac{1}{2}z(1-z)^2}.$$

$$\begin{aligned} \tfrac{1}{2} \int_0^1 z(1-z)^2 \, dz &= \tfrac{1}{2} \int_0^1 (z - 2z^2 + z^3) \, dz \\ &= \tfrac{1}{2} [\tfrac{1}{2}z^2 - \tfrac{2}{3}z^3 + \tfrac{1}{4}z^4]_0^1, \end{aligned}$$

so that the outer integral is

$$\boxed{\tfrac{1}{24}}.$$



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